

Proof of a Conjecture of Helleseeth: Maximal Linear Recursive Sequences of Period $2^{2^n} - 1$ Never Have Three-Valued Cross-Correlation

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Abstract

We prove a conjecture of Helleseeth that claims that for any $n \geq 0$, a pair of binary maximal linear sequences of period $2^{2^n} - 1$ can not have a three-valued cross-correlation function.

1 Introduction

The binary maximal linear sequences of period $2^m - 1$ are the sequences of elements in $\text{GF}(2)$ of the form $\{\text{Tr}(\alpha^{di+t})\}_{i \in \mathbb{Z}}$ where α is a generator of $\text{GF}(2^m)^*$, $\text{Tr}: \text{GF}(2^m) \rightarrow \text{GF}(2)$ is the absolute trace, and d and t are integers (or integers modulo $2^m - 1$) with $\gcd(d, 2^m - 1) = 1$. (See the Introduction of [2].) The cross-correlation of any two binary sequences $a = \{a_i\}$ and $b = \{b_i\}$ whose periods are divisors of $2^m - 1$ is the function $C_{a,b}(t) = \sum_{i=0}^{2^m-2} (-1)^{a_{i-t}+b_i}$. In this note, we shall take $a = \{a_i\} = \{\text{Tr}(\alpha^i)\}$ and $b = \{b_i\} = \{\text{Tr}(\alpha^{di})\}$, where the *decimation* d has $\gcd(d, 2^m - 1) = 1$. We call decimations with $d \equiv 1, 2, \dots, 2^{m-1} \pmod{2^m - 1}$ *trivial decimations* because $\{\text{Tr}(\alpha^{2^k i})\}$ is the same sequence as $\{\text{Tr}(\alpha^i)\}$. One readily shows that $C_{a,b}(t)$ is the same as

$$C_d(t) = \sum_{x \in \text{GF}(2^m)^*} (-1)^{\text{Tr}(\alpha^{-t}x + x^d)}.$$

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For a fixed d , we are interested in how many different values $C_d(t)$ takes as t varies over $\mathbb{Z}/(2^m - 1)\mathbb{Z}$. We say that $C_d(t)$ is v -valued to mean that $|\{C_d(t) : t \in \mathbb{Z}/(2^m - 1)\mathbb{Z}\}| = v$. Helleseth gave the following criterion for determining whether $C_d(t)$ is two-valued.

Theorem 1.1 (Helleseth [2], Theorem 3.1(d),(g), Theorem 4.1). *If $d \equiv 1, 2, \dots, 2^{m-1} \pmod{2^m}$, then $C_d(t) \in \{-1, 2^m - 1\}$ for all t . Otherwise, $C_d(t)$ takes at least three different values.*

In the same paper, Helleseth conjectured the following.

Conjecture 1.2 (Cf. Helleseth [2], Conjecture 5.2). *If m is a power of 2, $C_d(t)$ is not three-valued.*

In view of Theorem 1.1, this conjecture says that if m is a power of 2, then $C_d(t)$ is either two-valued (if d is a trivial decimation) or takes four or more values (if d is nontrivial). We prove this conjecture in this note.

Feng [1] recently proved the following weaker form of Conjecture 1.2.

Theorem 1.3 (Feng [1], Theorem 2). *If m is a power of 2 and $C_d(t) = -1$ for some value of t , then $C_d(t)$ cannot be three-valued.*

We prove Conjecture 1.2 by proving the following.

Theorem 1.4. *If $C_d(t)$ is three-valued, then $C_d(t) = -1$ for at least one value of t .*

This, combined with Theorem 1.3, immediately implies Conjecture 1.2.

Remark 1.5. One should note that our theorem does not assume m is a power of 2, so it is much more general in scope than what is needed. In fact, one can prove the same theorem for maximal linear sequences derived from fields $\text{GF}(p^m)$ with p odd: this (and more) is done in [3].

2 Proof of Theorem 1.4

We shall prefer to work in terms of the *Walsh transform*, defined as

$$W_d(a) = \sum_{x \in \text{GF}(2^m)} (-1)^{\text{Tr}(x^d + ax)},$$

and it is straightforward to show that

$$W_d(\alpha^{-t}) = 1 + C_d(t).$$

Thus the values of W_d on $\text{GF}(2^m)^*$ are just the values of C_d shifted by 1. So C_d is three-valued if and only if W_d is three valued on $\text{GF}(2^m)^*$.

We need to establish a few well-known facts before proceeding to the proof of Theorem 1.4. First, we need a simple result which, in rough terms, states that a sequence cannot be perfectly correlated or anti-correlated to a nontrivial decimation of itself.

Lemma 2.1. *If $d \not\equiv 1, 2, \dots, 2^{m-1} \pmod{2^m - 1}$, then $|W_d(a)| < 2^m$.*

Proof. From the definition of $W_d(a)$ as the sum $\sum_{x \in \text{GF}(2^m)} (-1)^{\text{Tr}(x^d + ax)}$ of 2^m terms in $\{1, -1\}$, it suffices to prove that the said terms are not all of the same sign. The $x = 0$ term is 1, and so the only way that all the terms can have the same sign is if

$$\text{Tr}(x^d + ax) = (x^d + x^{2d} + \dots + x^{2^{m-1}d}) + a(x + x^2 + \dots + x^{2^{m-1}})$$

equals 0 for all $x \in \text{GF}(2^m)$, i.e., if and only if this polynomial is zero modulo $x^{2^m} - x$. Given our assumption on d , all the exponents of x that appear in the polynomial as expressed above are distinct modulo $2^m - 1$, so this cannot happen. \square

We consider the first few power moments of W_d , with the r th power moment defined to be

$$P_r = \sum_{a \in \text{GF}(2^m)^*} W_d(a)^r,$$

where we use the convention $0^0 = 1$ in evaluating P_0 . The power moments of C_d have been calculated by Helleseeth, whence it is easy to obtain those of W_d .

Proposition 2.2 (See Helleseeth [2]). *We have*

- (a) $P_0 = 2^m - 1$,
- (b) $P_1 = 2^m$,
- (c) $P_2 = 2^{2m}$, and
- (d) $P_3 = 2^{2m}|V|$,

where V is the set of roots of $1 + x^d + (1 + x)^d$ in $\text{GF}(2^m)$.

From these one can readily deduce the following, which also appears as calculations in [1].

Proposition 2.3. *Suppose that $W_d(a)$ is three-valued on $\text{GF}(2^m)^*$ with values A , B , and C , and that $W_d(a) = C$ for N_C values of $a \in \text{GF}(2^m)^*$. Then*

$$N_C = \frac{2^{2m} - 2^m(A + B) + (2^m - 1)AB}{(C - A)(C - B)}$$

and

$$2^{2m}|V| = 2^{2m}(A + B + C) - 2^m(AB + BC + CA) + (2^m - 1)ABC,$$

where V is the set of roots of $1 + x^d + (1 + x)^d$ in $\text{GF}(2^m)$.

Proof. To get N_C , compute $\sum_{a \in \text{GF}(2^m)^*} (W_d(a) - A)(W_d(a) - B)$. On the one hand, $W_d(a) \in \{A, B, C\}$ implies that the sum is $N_C(C - A)(C - B)$. On the other hand, one can also calculate the sum in terms of power moments as $P_2 - (A + B)P_1 + ABP_0$, and then use the values given in Proposition 2.2. To get $|V|$, one can employ the same approach, this time with the sum $\sum_{a \in \text{GF}(2^m)^*} (W_d(a) - A)(W_d(a) - B)(W_d(a) - C)$: on the one hand, it is zero, and on the other, it can be expressed in terms of P_0 , P_1 , P_2 , and P_3 . \square

This can be used to prove an interesting result about the 2-divisibility of the values assumed by $W_d(a)$.

Lemma 2.4. *Suppose that $W_d(a)$ takes precisely three values A , B , and C for $a \in \text{GF}(2^m)^*$. If all three values are non-zero, then $2^{m+1} \mid AB$.*

Proof. From Proposition 2.3 we have

$$2^{2m}|V| = 2^{2m}(A + B + C) - 2^m(AB + BC + CA) + (2^m - 1)ABC, \quad (1)$$

where V is the set of roots of $1 + x^d + (1 + x)^d$ in $\text{GF}(2^m)$. Suppose that $A, B, C \neq 0$; then Lemma 2.1 shows that $A, B, C \not\equiv 0 \pmod{2^m}$. (We clearly have a nontrivial decimation by Theorem 1.1 since W_d is three-valued on $\text{GF}(2^m)^*$, and hence C_d is three-valued.) Then the term $(2^m - 1)ABC$ is divisible by fewer powers of 2 than the other terms on the right hand side of (1), so $2^{2m}|V|$ and ABC have exactly the same power of 2 in their respective prime factorizations, and so $2^{2m} \mid ABC$. Since $C \not\equiv 0 \pmod{2^m}$, this means that $2^{m+1} \mid AB$. \square

Now we are ready to prove Theorem 1.4. We assume that C_d is three-valued and that none of these values is -1 in order to show a contradiction. Then $W_d(a)$ is three-valued for $a \in \text{GF}(2^m)^*$ with the three nonzero values A , B , C . Note that Proposition 2.2(b) shows that

$$\sum_{a \in \text{GF}(2^m)^*} W_d(a) = 2^m,$$

so we cannot have $A, B, C < 0$. Furthermore, by parts (b) and (c) of the same proposition,

$$\left(\sum_{a \in \text{GF}(2^m)^*} W_d \right)^2 = 2^{2m} = \sum_{a \in \text{GF}(2^m)^*} W_d(a)^2,$$

so we cannot have $A, B, C > 0$. Then without loss of generality, we may take $A < 0 < B$ and C not between A and B . Then by Proposition 2.3, the number N_C of $a \in \text{GF}(2^m)^*$ such that $W_d(a) = C$ is

$$N_C = \frac{2^{2m} - 2^m(A + B) + (2^m - 1)AB}{(C - A)(C - B)}.$$

Since C is not between A and B , the denominator is positive, so

$$2^{2m} - 2^m(A + B) + (2^m - 1)AB > 0.$$

We use Lemma 2.1 and the fact that $A < 0$ and $B > 0$ to see that

$$2^{2m} - 2^m(-(2^m - 1) + 1) + (2^m - 1)AB > 0,$$

so that $AB > -2^{m+1}$. But by Lemma 2.4 and the fact that $A < 0 < B$, we have that $AB \leq -2^{m+1}$, which gives the contradiction that completes the proof of Theorem 1.4.

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References

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